

# Gauge Theory of Gravity and Spacetime\*

Friedrich W. Hehl

Univ. of Cologne and Univ. of Missouri, Columbia, MO  
hehl@thp.uni-koeln.de

16 April 2012 *file WuppertalWorkshop02.tex*

## Abstract

The advent of general relativity settled it once and for all that a theory of spacetime is inextricably linked to the theory of gravity. From the point of view of the gauge principle of Weyl and Yang–Mills–Utiyama, it became manifest around the 1960s (Sciama–Kibble) that gravity is closely related to the Poincaré group acting in Minkowski space. The gauging of this external group induces a Riemann–Cartan geometry on spacetime. If one generalizes the gauge group of gravity, one finds still more involved spacetime geometries. If one specializes it to the translation group, one finds a specific Riemann–Cartan geometry with teleparallelism (Weitzenböck geometry).

---

\*Based on an invited seminar, given at: *Towards a Theory of Spacetime Theories*, International Workshop, 21 to 23 July 2010, IZWT, Bergische Universität Wuppertal, Germany; Dennis Lemkuhl, Erhard Scholz, and Gregor Schieman (organizers).

## Contents

1. Apropos a theory of spacetime theories
2. Is the gauge idea the underlying principle for all interactions?
3. The gauging of the Poincaré group
4. Einstein's discussion of the transition from special to general relativity
5. Neutron interferometer experiments
6. Some geometric machinery: coframe and connection
7. More geometry: metric and orthonormal coframe
8. Dirac field  $\Psi(x)$  in Minkowski space in a non-inertial reference frame
9. Some results of the Lagrange–Noether formalism
10. Field equations of Sciama and Kibble
11. Quadratic Poincaré gauge theory of gravity (qPG)
12. Outlook

## 1 Apropos a theory of spacetime theories

In this workshop, we are supposed to move “Towards a theory of spacetime theories”. The idea seems to be that there are many spacetime theories around, the Riemannian spacetime theory in the framework of general relativity (GR), the Weitzenböck spacetime theory in teleparallelism approaches to gravity, the Riemann–Cartan spacetime theory withing the Poincaré gauge theory of gravity (PG), the superspace(time) theory within supergravity, the Weyl(–Cartan) spacetime theory within a gauge theory of the Weyl group, etc.. The list could be continued with spacetime theories emerging in quantization approaches to gravity where spacetime becomes mostly a discrete structure. There is a plethora of different spacetime theories around and it is hardly possible to view all of them from some kind of a unifying principle, let alone from one theory encompassing these spacetime theories as specific subcases.

Orientation in this seemingly chaotic landscape of spacetime theories can be provided by looking at the successful theories of our days that are able to predict and describe correctly fundamental phenomena occurring in nature. There is the standard model of particle physics, based on the Poincaré group (also known as inhomogeneous Lorentz group) and the internal groups  $SU(3)$ ,  $SU(2)$ ,  $U(1)$ . The Poincaré group is the group of motion in the Minkowski spacetime of special relativity (SR) and it classifies the particles according to their masses and their spins. The internal groups describe the strong and the electro-weak interactions by means of the respective gauge (or Yang–Mills) theory.

A book on the centennial of the discovery of SR was called [1]: “Special Reativity. Will it survive the next 100 years?” When I read this title in 2005, I thought for a moment that I must have been in a time machine and in reality I am living in 1905. Hadn’t SR already been superseded in 1915/16 by GR, I wondered? I pointed this out to the editors that this title looks anachronistic to me and is hardly appropriate for editors who both are known to subscribe to GR. It turned out that both wanted to ask whether SR survives *locally* as a valid theory. But they didn’t want to change the title since this fact was, as they told me, known to everybody anyway. I gave up since I realized that in a time when in the tabloid press a title is more for catching one’s attention than for spreading the truth also the scientific literature cannot stand aside.

But what was my point? Well, we all seem to agree that at least presently SR is universally valid locally *in a freely falling frame*. So far no deviations therefrom have been found. Only at very high accelerations, the principle of locality, inherent in SR, may need to be amended [2]. In any case, our march towards a theory of all spacetime theories has at least a definitive starting point.

But SR was superseded by GR? Yes, of course. The abstraction of a Minkowski space can only be uphold when gravitational effect can safely be neglected. If you measure Planck’s constant or the elementary charge by a conventional laboratory experiment, then this assumption is justified. But if you go down the stairs, you better don’t neglect gravity, otherwise you may fall downwards; or if you measure the deflection angle of a light ray gracing a star, you also better don’t neglect gravity. From the laboratory to at least the scale of the planetary system, at least, GR is in excellent agreement with experiment. On the galactic scale this is taken for granted by most physicist, but this is disputed by supporters of MOND, of TeVeS, of  $f(R)$ -theory, or of nonlocal gravity, for example, compare the presentations in [3]. Anyway, GR is mostly accepted for the global description of the cosmos and if the cosmological principle is assumed, namely homogeneity and isotropy of space, Einstein’s field equation predicts a Friedmann cosmos. The

cosmos started with the Big Bang and it is usually assumed to be equipped with a scalar inflationary field providing a sufficiently fast expansion. Needless to say that this framework is based on a number of extreme extrapolations.

The message is then that the Minkowski spacetime picture is substituted by the Riemannian one. But this doesn't rest on the same strong experimentally well-confirmed basis as the local presence of the Minkowski spacetime of SR.

## 2 Is the gauge idea the underlying principle for all interactions?

Since the advent of GR it was clear that a spacetime theory is inextricably linked to gravity. One cannot be understood without the other. Coming back to the topic of our workshop, it is then clear that gravity has to be considered in this general context willy nilly. Accordingly, a spacetime theory is at the same time, at least in some of its parts, a theory of gravity.

Let us then turn to gravity: Is GR all we have? Well, by some people GR is declared to be sacrosanct and you may touch it only by superimposing some abstract mathematical framework supposedly quantizing GR, see [4]. But practitioners of this method increasingly become aware that they have to amend the Hilbert–Einstein Lagrangian of the free gravitational field by non-Riemannian supplementary terms thereby dissolving to a certain extent the Riemannian structure they started with [5, 6, 7]. Hence alternatives to GR gain credibility even if GR is left fixed at first.

Is GR the only reasonable theory of gravity? No, it isn't. Already in 1956 Utiyama began to formulate gravity as a gauge theory, for a selection of classical papers, see [8]. The strong and electro-weak gauge theories are based on internal symmetry groups—mathematically semi-simple Lie groups—linked to conserved currents. The gauge idea basically requires that the *rigid* (or global) symmetry group related to the conserved current under consideration has to be made *local*; without giving up the invariance of the Lagrangian, this is only possible by the introduction of a gauge potential  $A = A_i dx^i$  (a covector) that transform under this group suitably; for each parameter of the group one needs one covector field. Thus, the group dictates the interaction emerging from that scheme: a new interaction is created from a conserved current via the (reciprocal) Noether theorem and the symmetry group attached to it.

In the standard model of particle physics all gauge groups are internal, that is,

they act in some internal space. In the original Yang–Mills theory, for example, it was the isospin space. But the gauge idea of *localizing a symmetry* does not seem to be restricted to internal groups. An external group affects by definition spacetime. If we have a conserved current and a corresponding group, nothing prohibits us to apply the gauge principle.

How does gravity come into this framework? The source of Newtonian gravity is the mass of a body. In classical physics, mass is a conserved quantity, as has been experimentally proved by Lavoisier. In SR mass conservation is no longer valid and is superseded by energy-momentum conservation, as has been most vividly demonstrated in Alamogordo in 1945. Clearly then, the Poisson equation controlling Newton’s gravitational potential  $\phi$ , namely  $\Delta\phi(\mathbf{r}, t) = 4\pi G\rho(\mathbf{r}, t)$ , with  $\Delta$  as the Laplace operator,  $G$  as the gravitational constant, and  $\rho$  as the mass density, has to be substituted by an equation that carries on its right-hand-side the energy density of matter (and/or radiation). However, according to SR, the energy density is the time-time component of the symmetric energy-momentum current  $t_{ij} = t_{ji}$  of matter (and/or radiation).

For an isolated physical system, the energy-momentum current  $t_{ij}$  is conserved:  $\partial_j t_i{}^j = 0$ . This is an expression of the fact that the action of the system is invariant under translations in time and space. Consequently, the conserved energy-momentum current together with the translation group  $T(4)$  acting in Minkowski space should underlie gravity. Since the translation group has four parameters, one describing a time translation and three describing space translations, we expect four potential one-forms  $\vartheta^\alpha$ , for  $\alpha = 0, 1, 2, 3$ . As we will see further down, this framework leads to a teleparallelism theory of gravity and back to a theory that is equivalent to GR for conventional (bosonic) matter. Accordingly, GR can be understood as a gauge theory of the translation group  $T(4)$ , which is an *external* group.

Ergo, all interactions, including gravity, are governed by gauge field theories. But let us now turn back to the history of the gauge idea:

### 3 The gauging of the Poincaré group

As we mentioned before, Utiyama [9] first attacked the problem of understanding gravity as a gauge theory by means of gauging the Lorentz group  $SO(1, 3)$ . In this way, Utiyama supposedly derived general relativity. However, the problematic character of his derivation is apparent. First of all, he had to introduce in an ad hoc way tetrads  $e_i{}^\alpha$  (or coframes  $\vartheta^\alpha = e_i{}^\alpha dx^i$ ), first holonomic (natural) and later

anholonomic (arbitrary) ones. Secondly, he has to assume the connection  $\Gamma_i^{\alpha\beta}$  of spacetime to be Riemannian, without any convincing argument.

But thirdly, perhaps the strongest reason, the current linked to the (homogeneous) Lorentz group is the *angular momentum current*  $\mathfrak{J}_{ij}{}^k = -\mathfrak{J}_{ji}{}^k$ , which is conserved,  $\partial_k \mathfrak{J}_{ij}{}^k = 0$ . However, as we have seen in the last section, gravity is coupled to the conserved and symmetric energy-momentum current  $t_{ik}$ . Accordingly, Einstein (1915) took in general relativity the symmetric energy-momentum current  $t_{ik}$  as the source of gravity in his field equation and *not* the angular momentum current. Hence Utiyama was not on the right track. Interestingly enough, in numerous publications even today, the Lorentz group is incorrectly thought of as gauge group of GR; usually the conserved current coupled to it is not even noticed.

This can be also viewed from the translational gauge group of gravity, at which we arrived above. In a Minkowski space, as in any Euclidean space, the group of motions consists of translations *and* rotations. In fact, the semidirect product of the translation group and the Lorentz group,  $T(4) \ltimes SO(1, 3)$ , is the Poincaré group  $P(1, 3)$  with its  $4 + 6$  parameters (and its  $4 + 6$  gauge potentials  $\vartheta^\alpha$  and  $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ , respectively). In a Euclidean or Minkowskian space the translations do not live alone, they are accompanied, in a nontrivial way, by the (Lorentz) rotations. Accordingly, since we find reasons to gauge the translations in a Minkowski spacetime, it is hardly avoidable to gauge also the rotations. If one has spinless matter, this argument may be skipped. However, if we have fermionic matter, its rotational behavior is closely linked to the translational behavior. Kibble, who was the first to gauge the Poincaré group [10], asks in the following way [11]:

“... Is it possible that starting from a theory with rigid symmetries and applying the gauge principle, we can recover the gravitational field? The answer turned out to be yes, though in a subtly different way and with an intriguing twist. Starting from special relativity and applying the gauge principle to its Poincaré-group symmetries leads most directly not precisely to Einstein’s general relativity, but to a variant, originally proposed by Élie Cartan, which instead of a pure Riemannian space-time uses a space-time with torsion. In general relativity, curvature is sourced by energy and momentum. In the Poincaré gauge theory, in its basic version, additionally torsion is sourced by spin.”

This is also the basic message of our seminar: Gauging an external group, here the Poincaré group, leads directly to a new geometry of spacetime, here the

Riemann–Cartan geometry of spacetime. To an external gauge group a certain geometry of spacetime is attached, the Minkowski space is deformed in accordance with the gauged symmetries. Moreover, without a conserved current, there can be no real gauge procedure in the sense of Weyl and Yang–Mills. If somebody tries to sell you a gauge theory without mentioning the associated conserved current, don’t believe her or him a word. Gauging the Weyl group *without* considering the scale current and gauging the conformal group *without* considering the conformal currents, are procedures that may lead to something, but certainly not to gauge theories à la Weyl–Yang–Mills, see the discussion in [8].

Often I heard the argument that gravity can have no relation to the translation group since GR takes place in a Riemannian space and therein the translations are an ill-defined concept since they are not integrable, for example. However, this argument rests on a misunderstanding. In a gauge approach, at the start of the procedure, that is, before the rigid symmetry is made local, we consider the gravity-free case. Accordingly, we are in Minkowski space where a translation is part of the group of motion. Only after we localized the symmetry, we lose the underlying Minkowski space, it gets deformed, and one has to reconstruct the emerging geometry. This is the radicality of the gauge principle: an interaction is created by a symmetry. The translation group  $T(4)$ , a subgroup of the Poincaré group  $P(1, 3)$ , which acts in a Minkowski space, creates the gravitational potential  $\vartheta^\alpha$ . The Lorentz subgroup  $SO(1, 3)$  creates another gravitational potential  $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$ , the consequences of which we will have to discuss.

## 4 Einstein’s discussion of the transition from special to general relativity

Before we turn to the subject of the gauging of the Poincaré group, we remind ourselves how Einstein “derived” gravity [12]. When Einstein developed GR, he could take a classical mass point with mass  $m$  as a starting point for his investigations. He studied its behavior in an accelerated reference system. Technically, in order to switch on acceleration, he transformed the original Cartesian coordinate system  $X^i$  to a curvilinear coordinate system  $x^i$ . Let us look at this in more detail. The points in the Minkowski space of SR can be described with the help of Cartesian coordinates  $X^i$ , with  $i = 0, 1, 2, 3$ . In these coordinates, the line element reads

$$ds^2 = (dX^0)^2 - (dX^1)^2 - (dX^2)^2 - (dX^3)^2 = o_{ij}dX^i \otimes dX^j, \quad (1)$$

with  $o_{ij} = \text{diag}(1, -1, -1, -1)$  and summation over repeated indices. The equation of motion of a *force-free* mass in an inertial frame  $K$ ,

$$\frac{d^2 X^k}{ds^2} = 0, \quad (2)$$

leads for the particle trajectory to a straight line with constant velocity.

The same motion, as viewed from the accelerated frame  $K'$ , can be derived by a transformation of (2) to curvilinear coordinates,

$$\frac{D^2 x^k}{Ds^2} := \frac{d^2 x^k}{ds^2} + \tilde{\Gamma}_{ij}{}^k \frac{dx^i}{ds} \frac{dx^j}{ds} = 0, \quad (3)$$

with the Riemannian connection (Christoffel symbols of the 2nd kind):

$$\tilde{\Gamma}_{ij}{}^k := \frac{1}{2} g^{k\ell} (\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}) = \tilde{\Gamma}_{ji}{}^k; \quad (4)$$

here we abbreviated the partial differentiation  $\partial/\partial x^i$  as  $\partial_i$ . The massive particle accelerates with respect to the non-inertial frame  $K'$  in such a way that this acceleration is independent of its mass. But an observer in  $K'$  cannot tell whether this motion is accelerated or induced by a homogeneous gravitational field of strength  $\tilde{\Gamma}_{ij}{}^k$ . In other words, the reference system  $K'$  can be alternatively considered as being at rest with respect to  $K$ , but a homogeneous gravitational field is present that is described by the *Christoffel symbols*  $\tilde{\Gamma}_{ij}{}^k$ .

Nothing has happened so far. We are still in a Minkowski space in which—as is shown in geometry—the *Riemann curvature tensor* belonging to the Christoffel symbols

$$\tilde{R}_{ijk}{}^\ell := 2\partial_{[i}\tilde{\Gamma}_{j]k}{}^\ell + 2\tilde{\Gamma}_{[i|m}{}^\ell \tilde{\Gamma}_{j]k}{}^m \quad (5)$$

vanishes, that is  $\tilde{R}_{ijk}{}^\ell = 0$ ; brackets around indices denote antisymmetrization:  $[ij] := \{ij - ji\}/2$ . This is the ingenuity of Einstein's approach: He considers force-free motion from two different reference frames and identifies thereby the Christoffels as describing—according to the equivalence principle—a homogeneous gravitational field. Of course, this gravitational field in Minkowski space is fictitious, it is simulated, it doesn't really exist since the Riemann curvature vanishes.

Besides massive point particles, we have light rays (“photons”) that can be considered in a similar way. For light propagation we have  $ds^2 = 0$ , but the geodesic line (3) can be reparametrized with the help of a suitable affine parameter. Then, from the point of view of reference frame  $K'$ , a light ray that propagates



in a straight line in the inertial frame  $K$  appears to be deflected in  $K'$ . According to Einstein [13], “...the principle of the constancy of the *velocity of light in vacuo* must be modified, since we easily recognize that the path of the light ray with respect to  $K'$  must in general be curvilinear.” Thus, the gravitational field deflects light. This is one of Einstein famous and successful predictions.

In order to create a real gravitational field—this is Einstein’s assumption—we must relax the rigidity of Minkowski space and allow for Riemannian curvature, inducing in this way a “deformed” spacetime carrying non-vanishing curvature  $\tilde{R}_{ijk}{}^\ell \neq 0$ . A prerequisite for this procedure to work is the fact that the Christoffels depend at most on first derivatives  $\partial_k g_{ij}(x)$  of the metric  $g_{ij}(x)$ . These first derivatives appear even in a flat space in an accelerated frame. Only non-vanishing second derivatives tell us about real gravitational fields.

There is one more thing to be seen from (3). If we multiply it with a slowly varying scalar mass density  $\rho$  of dust matter, then we recognize that the Christoffels are coupled to the (symmetric) energy-momentum tensor density of dust,<sup>1</sup>

$$\rho \frac{d^2 x^k}{ds^2} + \mathfrak{t}^{ij} \tilde{\Gamma}_{ij}{}^k = 0 \quad \text{with} \quad \mathfrak{t}^{ij} := \rho u^i u^j \quad (6)$$

and  $u^i := dx^i/ds$  as velocity of the dust. The fictitious non-tensorial force density  $\mathfrak{f}^k := \mathfrak{t}^{ij} \tilde{\Gamma}_{ij}{}^k$ , as observed by Weyl [15], is somewhat analogous to the Lorentz force acting on a charged particle in electrodynamics  $\mathfrak{f}_{\text{Lor}}^k := \mathfrak{J}^i F_i{}^k$ , with  $\mathfrak{J}^i = \rho_{\text{el}} u^i$  as electric current density and  $F_{ik}$  as electromagnetic field strength, the difference being that here the force density  $\mathfrak{f}^k$  is quadratic in  $u^i$ , whereas the Lorentz force density  $\mathfrak{f}_{\text{Lor}}^k$  is linear in  $u^i$ ; note also that the electromagnetic field is antisymmetric  $F_{ik} = -F_{ki}$  and the gravitational field symmetric  $\tilde{\Gamma}_{ij}{}^k = +\tilde{\Gamma}_{ji}{}^k$ . Thus, as a byproduct, we have identified the energy-momentum tensor density of matter as the source of gravity.

## 5 Neutron interferometer experiments

However, in the meantime, I mean since 1916, we have learned that there are fermions in nature. Besides mass  $m$ , they carry half-integer spin  $s$ . Instead of a mass point, we will then study the simplest massive fermion, the Dirac field in an inertial and a non-inertial reference frame thus taking care of Synge’s verdict

---

<sup>1</sup> A more detailed discussion can be found in Adler, Bazin, and Schiffer [14], p. 351.

“Newton successfully wrote *apple* = *moon*, but you cannot write *apple* = *neutron*”. This is what, in fact, Kibble [10] has done in 1961.

But even better, experimentally it has been clear since 1975 that the Colella–Overhauser–Werner (COW) experiment [16] is the “modern” archetypal experiment for a fermion in a gravitational field: A monochromatic neutron beam, extracted from a nuclear reactor, falls freely in the gravitational field of the earth. The phase shift of its wave function  $\Psi(x)$ , caused by the gravitational field, is measured by means of an interferometer built from a silicon mono-crystal, see also [17]. Accordingly, the single-crystal interferometer is at rest with respect to the laboratory, whereas the neutrons are subject to the gravitational potential. Bonse and Wroblewski (BW) [18] compared this with the effect of *acceleration* relative to the laboratory frame by letting the interferometer oscillate horizontally. With these experiments of BW and COW the effect of local acceleration and local gravity on matter waves has been shown to be equivalent. Later, with atomic beam interferometry, the accuracy of these type of results were appreciably improved.

It is strange, but in most textbooks on gravitation—and in most philosophical discussions on gravity—these successful experiments on the behavior of Dirac fields under acceleration (BW) and in a gravitational field (COW) are simply not mentioned. Most textbook authors and philosophers rather restrict themselves to Einstein’s 1916 discussion and to experiments related therewith. In writing a textbook on gravitation, is it indecent to refer to experiments that have a certain quantum flavor? Is it appropriate to be silent about experiments that provide new insight into the structure of the gravitational field?

The neutrons in the COW and BW experiments have spin  $\frac{1}{2}$ , they are fermions. At the energies prevalent in the COW and the BW experiments, the neutron (including its spin) can be supposed to be elementary, its composition out of three quarks can be neglected. Accordingly, if the neutron is force-free, it can be described by a Dirac spinor  $\Psi(x)$  obeying the free Dirac equation<sup>2</sup>  $(i\gamma^k \partial_k - m)\Psi(x) = 0$ . Thus, the neutron obeys approximately a classical one-particle equation, namely the Dirac or, in the non-relativistic limit, the Pauli-Schrödinger equation and, if the spin can be neglected, the Schrödinger equation. That this evaluation is correct has been borne out by experiments of the COW and BW type [17]: the neutrons of the COW and the BW experiments obey a Schrödinger equation including a Newtonian gravitational potential energy or a corresponding

---

<sup>2</sup>Here  $\hbar = 1$ ,  $c = 1$ , the imaginary unit is denoted by  $i$ , the Dirac gamma matrices by  $\gamma^k$ , and the mass of the neutron by  $m$ . If an electromagnetic field is present, the Dirac equation has to be coupled minimally to it and a Pauli-term added that takes into account the non-standard magnetic moment of the neutron.

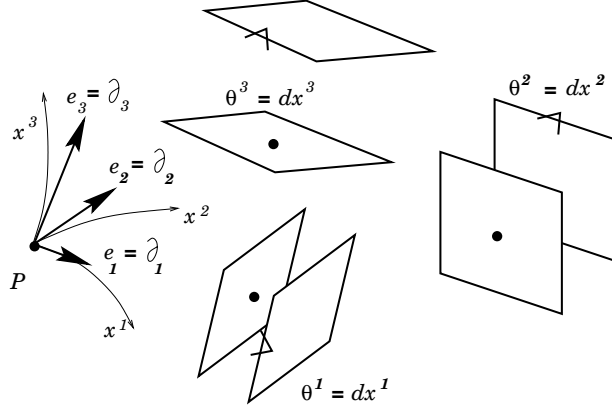


Figure 1: Natural frame  $e_b = \delta_b^j \partial_j$  and natural coframe  $\vartheta^a = \delta_i^a dx^i$  at a point  $P$  of a three-dimensional manifold ( $a, b = 1, 2, 3$ ). The coordinates of  $P$  are denoted by  $x^i$ ,  $i = 1, 2, 3$ , whereas  $\delta_a^b$  is the Kronecker symbol. The coframe  $\vartheta^a$  is supposed to be also at the same point  $P$ , but the three one-forms  $\vartheta^a$  are shifted for better visibility in 3 different directions. Note that  $\vartheta^1(e_1) = 1$ ,  $\vartheta^1(e_2) = 0$ , etc., that is,  $\vartheta^a$  is dual to  $e_b$  according to  $e_b \lrcorner \vartheta^a \equiv \vartheta^a(e_b) = \delta_b^a$ ; for the figure, see [19].

acceleration term, respectively.

The basic difference between the mass point and the Dirac field is that the latter requires an *orthonormal* reference frame for its description. A Dirac spinor is a half-integer representation of the [covering group  $SL(2, C)$  of the] Lorentz group  $SO(1, 3)$ , that is, it is intrinsically tied to the Lorentz group. In Minkowski space it is simple to introduce an orthonormal frame. One starts with Cartesian coordinates and takes the tangent vectors of the coordinate lines as “natural” frame  $e_\beta = \delta_\beta^j \partial_j$ , compare Figure 1. If one translates and Lorentz rotates such a frame, one can find an arbitrary frame  $e_\beta = e^j_\beta \partial_j$  that, in general, cannot any longer be derived from coordinate lines. But let us discuss this from a more general point of view.

## 6 Some geometric machinery: coframe and connection

Suppose that spacetime is a four-dimensional continuum in which we can distinguish one time and three space dimensions. At each point  $P$ , we can span the lo-

cal cotangent space by means of four *linearly independent* covectors, the *coframe*  $\vartheta^\alpha = e_i^\alpha dx^i$ . Here  $\alpha, \beta, \dots = 0, 1, 2, 3$  are frame and  $i, j, \dots = 0, 1, 2, 3$  coordinate indices. In general, the object of anholonomy two-form does not vanish,

$$C^\alpha := d\vartheta^\alpha = \frac{1}{2}C_{ij}^\alpha dx^i \wedge dx^j \neq 0, \quad \text{with} \quad C_{ij}^\alpha = 2\partial_{[i}e_{j]}^\alpha, \quad (7)$$

see [20]. This specification of spacetime is the bare minimum that one needs for applications to classical physics.

As soon as we have a coframe  $\vartheta^\alpha$ , we can also define its dual, the frame composed of four likewise linearly independent vectors  $e_\alpha = e^i_\alpha \partial_i$  by the duality relation  $e_\beta \lrcorner \vartheta^\alpha = \delta_\beta^\alpha$ . Geometrically speaking, frame and coframe are equivalent as reference frames for physical quantities. For physical reasons, the coframe turns out to be the translational gauge type potential and thus does fit more smoothly into a gauge formalism.

Having now a reference coframe  $\vartheta^\alpha$ , we want to do physics in such a space-time. We need a tool to express, for instance, that a certain field is constant. If the field is a scalar  $\phi$ , there is no problem, the gradient  $d\phi = (\partial_i \phi) dx^i$ , if equated to zero, will do the job. However, if the field is a vector or, more generally, a spinor or an arbitrary tensor field  $\psi$ , we need a law that specifies the parallel transfer of  $\psi$  from one point  $P$  to a neighboring point  $P'$ . Let us see how Einstein in 1955 looked in retrospect at the development of GR [21]:

...the essential achievement of general relativity, namely to overcome ‘rigid’ space (ie the inertial frame), is *only indirectly* connected with the introduction of a Riemannian metric. The directly relevant conceptual element is the ‘displacement field’ ( $\Gamma_{ik}^l$ ), which expresses the infinitesimal displacement of vectors. It is this which replaces the parallelism of spatially arbitrarily separated vectors fixed by the inertial frame (ie the equality of corresponding components) by an infinitesimal operation. This makes it possible to construct tensors by differentiation and hence to dispense with the introduction of ‘rigid’ space (the inertial frame). In the face of this, it seems to be of secondary importance in some sense that some particular  $\Gamma$  field can be deduced from a Riemannian metric...<sup>3</sup>

---

<sup>3</sup>When I showed this quotation during my seminar, E. Scholz (Wuppertal) immediately remarked that the fact of the importance of the connection as guiding field was already clear to Weyl in 1918, or at least in the 1920s. And D. Rowe (Mainz) added that also Einstein was aware of the importance of the concept of a connection since at least the late 1920s. Both remarks are certainly true. However, there is a subtle difference: Weyl referred to a symmetric connection since he was

Einstein’s ‘displacement field’ can be implemented by means of a *linear connection*  $\Gamma_\alpha^\beta = \Gamma_{i\alpha}^\beta dx^i$  (“affinity”). The one-form field  $\Gamma_\alpha^\beta(x)$ , with its 64 independent components, has to be prescribed before the parallel transport of a spinor or a tensor field  $\psi$  can be performed and, associated with it, a covariant derivative be defined (whose vanishing would imply that the field is constant). The linear connection  $\Gamma_\alpha^\beta(x)$ , shortly after the advent of general relativity, was recognized as a fundamental ingredient of spacetime physics, for more details see [8], for instance. The law of parallel transport embodies the *inertial properties* of matter.

The connection  $\Gamma_\alpha^\beta$  represents  $4 \times 4$  potentials of the four-dimensional group of general linear transformations  $GL(4, R)$ . Very similar to the Yang–Mills potential of the  $SU(3)$ , for example.

Coframe and connection  $\vartheta^\alpha, \Gamma_\alpha^\beta$ —still the metric is not involved—provide a good arsenal for further geometrical battles. Having a connection, we can covariantly differentiate. We define straightforwardly the “field strengths” torsion  $T^\alpha$  and curvature  $R_\alpha^\beta$  as

$$T^\alpha := d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta = \tfrac{1}{2} T_{ij}^\alpha dx^i \wedge dx^j, \quad (8)$$

$$R_\alpha^\beta := d\Gamma_\alpha^\beta - \Gamma_\alpha^\gamma \wedge \Gamma_\gamma^\beta = \tfrac{1}{2} R_{ij\alpha}^\beta dx^i \wedge dx^j. \quad (9)$$

One recognizes that  $T^\alpha$  and  $R_\alpha^\beta$  are the gauge field strengths of the affine group  $A(4, R) = T(4) \rtimes GL(4, R)$ .

Let us look at the torsion in components. From (8) we find

$$T_{ij}^\alpha = 2\partial_{[i} e_{j]}^\alpha + 2\Gamma_{[i|\beta}^\alpha e_{|j]}^\beta = C_{ij}^\alpha + 2\Gamma_{[ij]}^\alpha. \quad (10)$$

In a holonomic (coordinate) frame,  $C_{ij}^\alpha = 0$ . Thus,  $T_{ij}^\alpha \stackrel{*}{=} 2\Gamma_{[ij]}^\alpha$ . In such a frame—and only in a holonomic one—the vanishing of the torsion translates into the *symmetry of the connection*. It is now obvious why this symmetry is called a “bastard symmetry”: in  $\Gamma_{[ij]}^\alpha = \Gamma_{[i|\beta}^\alpha e_{|j]}^\beta$ , the index ‘i’ originates from the one-form character of the connection, whereas the index ‘j’ is related to the Lie-algebra index ‘ $\beta$ ’. Only in a holonomic frame the symmetry of a connection looks natural. In an anholonomic frame, here  $C_{ij}^\alpha \neq 0$ , it is nothing trivial. It is a fundamental assumption that has to be justified similar as the vanishing of the curvature.

---

concerned with coordinates and not with frames. When, in 1929 he introduced frames [22], Weyl’s connection still remained symmetric, and only in 1950 he considered also *asymmetric* connections in the context of gravity [23]. In contrast, Einstein was concerned with *asymmetric* connections at least since 1925, when he formulated a unified theory of gravity and electricity and introduced what is nowadays called incorrectly the Palatini variational principle [24].

A space with  $T^\alpha \neq 0$ ,  $R_\alpha^\beta \neq 0$ , we call an *affine* space. If  $T^\alpha = 0$ , we have a *symmetric* affine space, if  $R_\alpha^\beta = 0$ , we have a *teleparallel* affine space (or of a space with teleparallelism). Should we require  $T^\alpha = 0$  and  $R_\alpha^\beta = 0$ , we have a symmetric flat affine space.

We followed here the lead of Schrödinger [25] and introduced first the connection before we will turn to the metric.

## 7 More geometry: metric and orthonormal coframe

However, our experience in Minkowski space tells us that there must be more structure on the spacetime manifold than the symmetric flat affine space possesses. Locally at least, we are able to measure time and space intervals and angles. A pseudo-Riemannian (or Lorentzian) metric<sup>4</sup>  $g_{ij} = g_{ji}$  is sufficient for accommodating these measurement procedures. If  $g_{\alpha\beta}$  denotes the components of the metric with respect to the coframe, we have  $g_{ij} = e_i^\alpha e_j^\beta g_{\alpha\beta}$  and  $\mathbf{g} = g_{\alpha\beta} \vartheta^\alpha \otimes \vartheta^\beta$ . In an orthonormal coframe we recover

$$g_{\alpha\beta} \stackrel{*}{=} o_{\alpha\beta} := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \quad (11)$$

Now, in analogy to the procedures in equations (8) and (9), we can derive the field strength, the *nonmetricity* one-form, corresponding to the potential  $g_{\alpha\beta}$ , by differentiation:

$$Q_{\alpha\beta} := -Dg_{\alpha\beta} = -dg_{\alpha\beta} + \Gamma_\alpha^\gamma g_{\gamma\beta} + \Gamma_\beta^\gamma g_{\alpha\gamma} = Q_{i\alpha\beta} dx^i. \quad (12)$$

Accordingly, the coframe  $\vartheta^\alpha(x)$ , the linear connection  $\Gamma_\alpha^\beta(x)$ , and the metric  $g_{\alpha\beta}(x)$  control the geometry of spacetime. The metric determines the distances and angles, the coframe serves as translational gauge potential, whereas the connection provides the guidance field for matter reflecting its inertial properties and it is the  $GL(4, R)$  gauge potential. The space equipped with these  $10 + 16 + 64$

---

<sup>4</sup>Nowadays there exists a definite hint that the conformally invariant part of the metric, the light cone, is electromagnetic in origin (see [19, 26]), that is, it can be derived from premetric electrodynamics together with a linear constitutive law for the empty spacetime (vacuum). Hence the metric, or at least its conformally invariant part, doesn't appear as a fundamental structure, it rather emerges in an electromagnetic context.

potentials  $(g_{\alpha\beta}, \vartheta^\alpha, \Gamma_\alpha{}^\beta)$  we call a *metric-affine* space, the corresponding field strength are the  $40 + 24 + 96$  fields  $(Q_{\alpha\beta}, T^\alpha, R_\alpha{}^\beta)$ , for reviews and the corresponding formalism, see [8, 27, 28].

In a metric-affine space, we can lower the second index of the connection according to  $\Gamma_{\alpha\beta} := \Gamma_\alpha{}^\gamma g_{\gamma\beta}$ . Then we can compare it with the Riemann (Levi-Civita) connection  $\tilde{\Gamma}_{\alpha\beta}$ . After some algebra, see [20], we find in terms of components:

$$\Gamma_{\alpha\beta\gamma} = \tilde{\Gamma}_{\alpha\beta\gamma} + \frac{1}{2}(T_{\alpha\beta\gamma} - T_{\beta\gamma\alpha} + T_{\gamma\alpha\beta}) + \frac{1}{2}(Q_{\alpha\beta\gamma} + Q_{\beta\gamma\alpha} - Q_{\gamma\alpha\beta}). \quad (13)$$

It should be stressed that this decompositions are useful if a direct comparison is made with the Riemannian piece  $\tilde{\Gamma}$ . However, in the variational formalism of a gauge theory of gravity, besides  $g_{\alpha\beta}$  and  $\vartheta^\alpha$ , the connection  $\Gamma_\alpha{}^\beta$  is considered as *independent* variable. Then such a decomposition is unwarranted under those circumstances.

Can we give a satisfactory justification for the emergence of three different gravitational gauge potentials? We take the Minkowski space of SR as basis for our considerations. It is a fact of life that the geometry of a Minkowski (or a Euclidean) space consists of an interplay between properties that relate to parallel displacement and those that relate to distance and angle measurements. In Minkowski space this duality between affine (inertial) and metric properties is solved in that the affine properties are exclusively expressed in terms of metric properties: the metric properties dominate the affine ones.

If we “liberate” the affine properties, we are immediately led, in four dimensions, to the affine group  $A(4, R) = T(4) \rtimes GL(4, R)$  and, gauging it, to the coframe  $\vartheta^\alpha$  and the linear connection  $\Gamma_\alpha{}^\beta$  as gauge potentials. The metric properties, expressed by the metric  $g_{ij}$ , are then left behind.

Since macroscopic gravity in GR is so successfully described by means of the metric  $g_{ij}$  as (Einstein’s) gravitational potential, it suggests itself to add the metric—in its anholonomic form  $g_{\alpha\beta}$ —as third member to the gravitational potentials. There are two procedures possible: We transform the coframe, using the metric, to an *orthonormal coframe*; in this way the metric is absorbed and, besides this coframe, only the connection remains as variable. However, since this restricts the freedom of choosing also non-orthonormal coframes, we take all three potentials as independent variables. The Lagrangian formalism of the corresponding field theory will then provide the relation between the coframe and the metric, and it will turn out that there is, indeed, a close link between both variables, see [28]. At the same time—and this is a real progress in understanding—we find that

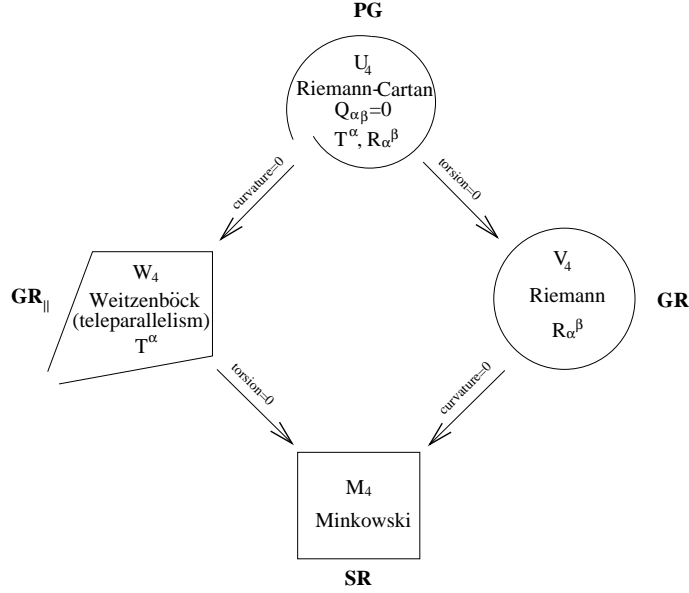


Figure 2: The Riemann–Cartan space (or  $U_4$ ), a metric-affine space with vanishing nonmetricity, is the arena for the Poincaré gauge theory (PG). It can either become a *Weitzenböck* space  $W_4$ , if its curvature vanishes, or a *Riemann* space  $V_4$ , if the torsion happens to vanish. GR acts in a  $V_4$ , teleparallelism theories of gravity in a  $W_4$ .

the metric energy-momentum current of matter  $t^{\alpha\beta}$  couples to the metric and the canonical one  $\mathfrak{T}_\alpha$  couples to the coframe. Their interdependence is beautifully displayed in the three-potentials’ approach.

In a metric-affine space, as shown by Hartley [29], *normal frames* can be found: *locally* it is possible to find suitable coordinates and suitable frames such that

$$(\vartheta^\alpha, \Gamma_\alpha^\beta) \stackrel{*}{=} (\delta_i^\alpha dx^i, 0). \quad (14)$$

This is the new type of Einstein elevator. In GR, the Einstein elevator was described by a holonomic reference frame  $\vartheta^\alpha$  with  $C^\alpha = 0$ . Then, in the Riemann spacetime of GR, one could introduce Riemannian normal coordinates. Here, in the gauge theoretical approach, the constraint of holonomicity is dropped and this new degree of freedom, which expresses itself in a rotational acceleration, admits to introduce normal frames. The equivalence principle can then be applied in this new context. For new developments of this notion, see Nester [30] and Giglio &



Rodrigues [31].

As soon as we require in a metric-affine space integrability of length and angle measurements, we have to postulate<sup>5</sup>  $Q_{\alpha\beta} = 0$ . Then we arrive at a *Riemann–Cartan* space (RC-space), which was mentioned in the context of the gauging of the Poincaré group in Sec. 3. In such a space, if we choose *orthonormal* frames, the connection becomes antisymmetric; then we have the  $6 + 24$  potentials  $(\vartheta^a, \Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha})$  as gravitational variables. Again normal frames like in (14) can be found,

$$(\vartheta^\alpha, \Gamma^{\alpha\beta}) \stackrel{*}{=} (\delta_i^\alpha dx^i, 0), \quad (15)$$

as has been first shown by von der Heyde [33]. This geometrical fact shows clearly that the Lorentz connection  $\Gamma^{\alpha\beta}$  is, besides the orthonormal frame  $\vartheta^\alpha$ , the appropriate gauge field variable. After this geometrical detour we are back to where we started from. In Figure 2 different subcases of a RC-space are displayed.

## 8 Dirac field $\Psi(x)$ in Minkowski space in a non-inertial reference frame

After this rather long geometrical interlude, we come back to physics and consider again the Dirac field  $\Psi(x)$ . A mass point in an inertial frame moves according to equation (2), one in an accelerated frame, according to equation (3). The inertial forces are represented by the Christoffels in equation (4). Let us execute the analogous process for the Dirac electron. Since the Dirac electron is referred to an orthonormal (co)frame, we have to study its behavior under translational and rotational accelerations, see [34].

In Minkowski space in Cartesian coordinates, we have the force-free Dirac equation as analog of equation (2),

$$(i\gamma^i \partial_i - m)\Psi \stackrel{*}{=} 0, \quad (16)$$

and in a non-inertial frame in flat Minkowski space we find

$$\left[ i\gamma^\alpha e^i_\alpha (\partial_i + \frac{i}{4} \sigma_{\beta\gamma} \tilde{\Gamma}_i^{\beta\gamma}) - m \right] \Psi = 0, \quad \sigma_{\beta\gamma} := i\gamma_{[\beta}\gamma_{\gamma]}. \quad (17)$$

---

<sup>5</sup>If one wants to keep the angles integrable, but not the length, one can postulate only the vanishing of the tracefree part of the nonmetricity,  $Q_{\alpha\beta} - \frac{1}{4}g_{\alpha\beta}Q_\gamma{}^\gamma = 0$ . This results in a Weyl–Cartan space with non-vanishing Weyl covector  $\frac{1}{4}Q_\gamma{}^\gamma$ , see the contribution of Scholz [32]; however, in this approach also the torsion is put to zero.

These two equations correspond to the Einsteinian equations (2) and (3). Namely, in the non-relativistic WKB-approximation, when the spin can be neglected, equation (16) becomes (2). You may wonder whether this is true since (16), in contrast to (2), is mass dependent. For this reason some people argued that this violates the equivalence principle in the sense that the motion of a force-free particle (field) must be independent of  $m$ . However, what they overlooked is that also in classical mechanics the Hamilton-Jacobi equation for a force-free particle *is* mass dependent—and the classical non-relativistic analog of the Dirac equation is the Hamilton-Jacobi equation. Accordingly, all is fine and in the desired approximation the mass will drop out.

The new potentials, emerging in a non-inertial frame, are  $(e^i_\alpha, \tilde{\Gamma}_i^{\beta\gamma})$ . The latter one, in Minkowski space, can be expressed in terms of derivatives of the former:  $\tilde{\Gamma}_i^{\beta\gamma} = \tilde{\Gamma}_i^{\beta\gamma}(\partial_j e^k_\delta)$ . However, we will not substitute  $\tilde{\Gamma}_i^{\beta\gamma}$  in terms of the frame since we will relax the constraint  $T_{ij}^\alpha = 0$  subsequently.

This is what we will do now. Einstein relaxed the constraint  $\tilde{R}_{ijk}^\ell = 0$ , since that is all he found for a point particle, we relax the constraints  $T_{ij}^\alpha = 0$  and  $\tilde{R}_{ij}^{\alpha\beta} = 0$ , since a Dirac field has a more involved structure as displayed in particular in a non-inertial frame. This relaxation of both constraints leads directly to a *Riemann–Cartan spacetime* as the arena appropriate for a Poincaré gauge theory (PG).

Why couldn't we do by only relaxing the curvature constraint,  $\tilde{R}_{ij}^{\alpha\beta} \neq 0$ , but keeping the torsion constraint,  $T_{ij}^\alpha = 0$ ? Well, this is possible. However, it is not in the sense of local field theory. Why should we keep the non-local constraint  $T_{ij}^\alpha = 0$ , which corresponds to 24 partial differential equations of first order, when we know that its relaxation does away with these PDEs and still allows locally to get rid of gravity according to (15)?

Whereas Einstein discussed the equivalence principle on the level of the equations of motion, in gauge theories, because of the application of the Noether theorem for rigid and local symmetries, the discussion takes place on the level of Lagrangians. If we multiply  $D_\alpha \Psi = (\partial_\alpha + \frac{i}{4}\sigma_{\beta\gamma}\Gamma_\alpha^{\beta\gamma})\Psi$  from the left by  $i\bar{\Psi}\gamma^\alpha$ , average with its Hermitian conjugate, and add a mass term, we find the (real) Dirac Lagrangian density:<sup>6</sup>

$$\begin{aligned} \frac{\mathcal{L}}{e} &= \frac{i}{2}e^i_\alpha \left[ \bar{\Psi}\gamma^\alpha \left( \partial_i + \frac{i}{4}\sigma_{\beta\gamma}\Gamma_i^{\beta\gamma} \right) \Psi \right] + \text{herm. conj.} + m\bar{\Psi}\Psi \\ &= \frac{i}{2}(\bar{\Psi}\gamma^\alpha D_\alpha \Psi - \Psi\gamma^\alpha \overline{D_\alpha \Psi}) + m\bar{\Psi}\Psi. \end{aligned} \quad (18)$$

---

<sup>6</sup> $e := \det e^i_\alpha$ ,  $\partial_\alpha = e^i_\alpha \partial_i$ ,  $D_\alpha = e^i_\alpha D_i$ .

The action is  $W = \int d^4x \mathcal{L}$ . The Lagrangian in an inertial frame in Cartesian coordinates can be read off by making the substitutions  $e^i{}_\alpha \rightarrow \delta^i_\alpha$ ,  $\Gamma_i{}^{\beta\gamma} \rightarrow 0$ .

## 9 Some results of the Lagrange–Noether formalism

To identify the currents that couple to the gravitational potentials  $(e_i{}^\alpha, \Gamma_i{}^{\alpha\beta})$ , some formalism is necessary that may disturb the philosophical minded reader. We try to simplify these considerations and will, instead of working in a RC-spacetime (for a rigorous treatment see [28]), restrict ourselves to the Minkowski space in Cartesian coordinates.

The action  $W$  is invariant under 4 rigid spacetime *translations* of 6 rigid Lorentz *rotations* (3 boosts plus 3 spatial rotations). As a consequence, we have (see Corson [35]) energy-momentum and angular momentum conservation,

$$\partial_k \mathfrak{T}_i{}^k \stackrel{*}{=} 0, \quad \partial_k \mathfrak{J}_{ij}{}^k \stackrel{*}{=} 0, \quad (19)$$

with the *canonical energy-momentum* tensor density

$$\mathfrak{T}_i{}^k \stackrel{*}{=} \delta_i^k \mathcal{L} - \frac{\partial \mathcal{L}}{\partial \partial_k \Psi} \partial_i \Psi, \quad (20)$$

the total canonical angular momentum tensor density, consisting of an intrinsic and an orbital part,

$$\mathfrak{J}_{ij}{}^k \stackrel{*}{=} \mathfrak{S}_{ij}{}^k + x_i \mathfrak{T}_j{}^k - x_j \mathfrak{T}_i{}^k = -\mathfrak{J}_{ji}{}^k, \quad (21)$$

and the *canonical spin* angular momentum tensor density ( $l_{ij}$ =Lorentz generators)

$$\mathfrak{S}_{ij}{}^k \stackrel{*}{=} \frac{\partial \mathcal{L}}{\partial \partial_k \Psi} l_{ij} \Psi = -\mathfrak{S}_{ji}{}^k. \quad (22)$$

From this straightforward consideration in Minkowski space alone, we recognize that the canonical energy-momentum  $\mathfrak{T}_i{}^k$  and the canonical angular momentum  $\mathfrak{J}_{ij}{}^k$  are the translational and the Lorentz currents of matter. Only the intrinsic spin part  $\mathfrak{S}_{ij}{}^k$  of the angular momentum is a tensor; the orbital part is only a tensor under Cartesian coordinate transformations. For the Dirac field we find ( $\mathfrak{T}_\alpha{}^k = e^i{}_\alpha \mathfrak{T}_i{}^k$ , etc.):

$$\mathfrak{T}_\alpha{}^i \stackrel{*}{=} \frac{i}{2} (\bar{\Psi} \gamma^i \partial_\alpha \Psi - \Psi \gamma^i \partial_\alpha \bar{\Psi}), \quad (23)$$

$$\mathfrak{S}_{\alpha\beta}{}^i \stackrel{*}{=} \frac{1}{8} \bar{\Psi} (\sigma_{\alpha\beta} \gamma^i + \gamma^i \sigma_{\alpha\beta}) \Psi. \quad (24)$$

The spin is totally antisymmetric: after some algebra, we can put (24) into the form

$$\mathfrak{S}_{\alpha\beta\gamma} = \frac{1}{4}\epsilon_{\alpha\beta\gamma\delta}\bar{\Psi}\gamma^\delta\gamma_5\Psi, \quad (25)$$

with  $\gamma_5 := -\frac{i}{4!}\epsilon_{\alpha\beta\gamma\delta}\gamma^\alpha\gamma^\beta\gamma^\gamma\gamma^\delta$ .

We compare (23) and (24) with the Lagrangian (18) and consider small deviations from the inertial case, that is,  $e^i_\alpha = \delta^i_\alpha + \epsilon^i_\alpha$ , with  $\epsilon^i_\alpha \ll 1$ , then we find after some algebra and some rearrangements to linear order in  $\epsilon^i_\alpha$ ,

$$\mathfrak{L} \sim e_i^\alpha \mathfrak{T}_\alpha^i + \Gamma_i^{\alpha\beta} \mathfrak{S}_{\alpha\beta}^i - m\bar{\Psi}\Psi. \quad (26)$$

There is some resemblance to the structure in (6) even though we work here on a Lagrangian level. This coupling of geometry to matter displayed in (26) suggests the following representation of the canonical currents:

$$\mathfrak{T}_\alpha^i = \frac{\delta\mathfrak{L}}{\delta e_i^\alpha}, \quad \mathfrak{S}_{\alpha\beta}^i = \frac{\delta\mathfrak{L}}{\delta \Gamma_i^{\alpha\beta}}. \quad (27)$$

Of course, this was a heuristic consideration, but with the full Lagrange-Noether machinery acting in RC-spacetime, it can be made rigorous [28]: The canonical currents  $\mathfrak{T}_\alpha^i$ ,  $\mathfrak{S}_{\alpha\beta}^i$ , defined via the Noether theorem according to (20) and (22), can be shown to be equal to the “dynamical” currents that couple to the gravitational potentials according to (27).

#### *A short summary of the formalism in this section*

For those of you who were lost in this formalism, a short bird eye’s view on the results: In order to compactify our notation, we change to exterior calculus. We introduce the matrix-valued one-form  $\gamma := \gamma_\alpha \vartheta^\alpha$  and the Hodge star operator  $*$ . Then the Dirac equation in an arbitrary orthonormal frame in a RC-space can be rewritten as

$$i^*\gamma \wedge D\Psi + *m\Psi = 0, \quad (28)$$

with the covariant exterior derivative  $D\Psi := (d + \frac{i}{4}\sigma_{\alpha\beta}\Gamma^{\alpha\beta})\Psi$ . Let us then formulate Lagrange four-form of the Dirac field,

$$L = L(\vartheta^\alpha, \Psi, D\Psi) = \frac{i}{2} (\bar{\Psi}^* \gamma \wedge D\Psi + \overline{D\Psi} \wedge * \gamma \Psi) + *m\bar{\Psi}\Psi, \quad (29)$$

which is minimally coupled to the RC-spacetime via the gauge potentials  $\vartheta^\alpha$  [contained in  $\gamma = \gamma_\alpha \vartheta^\alpha$ ] and  $\Gamma^{\alpha\beta} = -\Gamma^{\beta\alpha}$  [contained in  $D$ ]. Note that only the

potentials themselves enter the Lagrangian, but not their derivatives. Thus, the Lagrangian (29), formulated in a RC-spacetime, because of (15), looks *locally* special-relativistic. This attests to the validity of the relaxation process discussed above. The currents, as we saw above in (27), are then defined as follows:

$$\mathfrak{T}_\alpha = \frac{\delta L}{\delta \vartheta^\alpha}, \quad \mathfrak{S}_{\alpha\beta} = \frac{\delta L}{\delta \Gamma^{\alpha\beta}}. \quad (30)$$

These innocently looking equations (29) and (30), all living in a RC-spacetime, are the net outcome of our considerations so far.

It was then Sciama [36] and Kibble [10] in the early 1960s who added the Hilbert–Einstein type Lagrangian of the RC-spacetime to (18) and formulated the corresponding simplest field equations of the gauge theory of gravity; for a historical view see O’Raifeartaigh [37] and the reprint volume [8], for a modern representation Blagojević [38] and Ryder [39].

## 10 Field equations of Sciama and Kibble

The Ricci tensor in a RC-spacetime is defined according to  $\text{Ric}_i^\alpha := e^j_\beta R_{ji}^{\alpha\beta}$ . A corresponding scalar density  $ee^i_\alpha \text{Ric}_i^\alpha$  is the simplest nontrivial gravitational Lagrangian. The total action is ( $\Lambda$  = cosmological constant)

$$W_{\text{tot}} = \int d^4x \left[ \frac{1}{2\kappa} e(e^i_\alpha \text{Ric}_i^\alpha - 2\Lambda) + \mathfrak{L}(e_k^\gamma, \Psi, D\Psi) \right], \quad (31)$$

with Einstein’s gravitational constant  $\kappa$ . Variation with respect to  $e_i^\alpha$  and  $\Gamma_i^{\alpha\beta}$  yields the gravitational field equations of Sciama [36] and Kibble [10]:

$$\text{Ric}_\alpha^i - \frac{1}{2} e^i_\alpha \text{Ric}_\gamma^\gamma + \Lambda e^i_\alpha = \frac{\kappa}{e} \mathfrak{T}_\alpha^i, \quad (32)$$

$$\text{Tor}_{\alpha\beta}^i - e^i_\alpha \text{Tor}_{\beta\gamma}^\gamma + e^i_\beta \text{Tor}_{\alpha\gamma}^\gamma = \frac{\kappa}{e} \mathfrak{S}_{\alpha\beta}^i. \quad (33)$$

We made here the torsion a bit more visible. Please note that Ric and  $\mathfrak{T}$  have both 16 independent components, whereas Tor and  $\mathfrak{S}$  have both 24 independent components. These field equations are just linear algebraic equations between Ric and Tor on the geometrical side and  $\mathfrak{T}$  and  $\mathfrak{S}$  on the matter side, respectively. The Dirac case is particularly simple, there (33) collapses to just 4 equations.

The first equation can be easily recognized as an Einstein type field equation. However, the Ricci tensor is here asymmetric as well as the canonical energy-momentum tensor of matter. The second equation relates the torsion linearly to

the spin of matter. If we consider matter *without* spin, the torsion vanishes and the first field equation reduces just to the Einstein field equation of GR, for a review see [40].

In exterior calculus, these field equations, given first in this form by Trautman, see [41], look even a bit more transparent:<sup>7</sup>

$$\frac{1}{2}\eta_{\alpha\beta\gamma} \wedge R^{\beta\gamma} - \Lambda\eta_{\alpha} = \kappa \mathfrak{T}_{\alpha}, \quad (34)$$

$$\frac{1}{2}\eta_{\alpha\beta\gamma} \wedge T^{\gamma} = \kappa \mathfrak{S}_{\alpha\beta}. \quad (35)$$

The two equations (32),(33) or (34),(35) are the field equations of the Einstein–Cartan(–Sciama–Kibble) theory of gravity or, in short, of the *Einstein–Cartan theory* (EC). This is a special case of a Poincaré gauge theory, namely that which has the curvature scalar of the RC-spacetime as gravitational Lagrangian. EC is a viable gravitational theory.

The Maxwell field carries helicity, that is, spin projected along its wave vector, but it doesn’t carry spin proper as a gauge covariant quantity. Therefore, there is no electromagnetic contribution to the material spin on the right-hand-side of (33) or (35). Light is insensitive to torsion; torsion cannot be “seen”.

Torsion effects in EC-theory are minute. Besides the Einsteinian gravitational field, we have additionally a very weak *spin-spin contact interaction* that is proportional to the gravitational constant, which is measurable in principle. For a particle of mass  $m$  and reduced Compton wave length  $\lambda_{\text{Co}} := \hbar/mc$  (with  $\hbar$  = reduced Planck constant,  $c$  = speed of light), there exists in EC a critical density and, equivalently, a critical radius of ( $\ell_{\text{Pl}}$  = Planck length)

$$\rho_{\text{EC}} \sim m/(\lambda_{\text{Co}}\ell_{\text{Pl}}^2) \quad \text{and} \quad r_{\text{EC}} \sim (\lambda_{\text{Co}}\ell_{\text{Pl}}^2)^{1/3}, \quad (36)$$

respectively, see [40]. For a nucleon we have  $\rho_{\text{EC}} \approx 10^{54} \text{ g/cm}^3$  and  $r_{\text{EC}} \approx 10^{-26} \text{ cm}$ . Whereas those densities are extremely high from a usual lab perspective or even from the point of view of a neutron star ( $\approx 10^{16} \text{ g/cm}^3$ ), in cosmology they are standard. It may be sufficient to recall that inflation is believed to set in around the Planck density of  $10^{93} \text{ g/cm}^3$ .

At densities higher than  $\rho_{\text{EC}}$ , EC-theory is expected to overtake GR. There is no reason why GR should survive under those conditions, since for fermions the gauge-theoretical framework is much more trustworthy. Some cosmological models of EC can be found in [8].

---

<sup>7</sup>Here we have: Hodge star  $\star$ ,  $\eta_{\alpha} = \star\vartheta_{\alpha}$ ,  $\eta_{\alpha\beta} = \star(\vartheta_{\alpha} \wedge \vartheta_{\beta})$ ,  $\eta_{\alpha\beta\gamma} = \star(\vartheta_{\alpha} \wedge \vartheta_{\beta} \wedge \vartheta_{\gamma})$ . Moreover,  $\mathfrak{T}_{\alpha} = \frac{1}{e}\mathfrak{T}_{\alpha}{}^{\gamma}\eta_{\gamma}$  and  $\mathfrak{S}_{\alpha\beta} = \frac{1}{e}\mathfrak{S}_{\alpha\beta}{}^{\gamma}\eta_{\gamma}$ .

It is probably fair to say that EC has been established as a consistent and viable theory of gravity and the Riemann–Cartan geometry of spacetime has won solid support so that its study should not be skipped in philosophical circles as undesirable complication of the Riemann geometry of GR.

## 11 Quadratic Poincaré gauge theory of gravity (qPG)

Let me first express a word of caution: In a fairly recent paper, Mao, Tegmark, Guth, and Cabi [42] believe to have shown “...that Gravity Probe B is an ideal experiment for further constraining nonstandard torsion theories,...” Nothing could be further away from the truth. Following the guiding principle that nothing is more practical than a good theory, Puetzfeld and Obukhov [43] have shown that the measurement of torsion requires elementary particle spins as test objects whereas in Gravity Probe B the rotating quartz balls carry orbital angular momentum only, but don’t carry uncompensated elementary particle spin. Thus, the results in [42] are simply incorrect in spite of the wide publicity that this paper has won.

But back to Einstein–Cartan theory (EC). It is in many ways a very degenerate theory. A contact interaction in physics cries for a generalization to a propagating interaction, as has been the way things developed in the Fermi theory of weak interaction—which was a contact interaction par excellence—to the theory of the propagating  $W$  and  $Z$ . The recipe is very simple: The EC-Lagrangian is linear in the Lorentz field strength, add terms that are quadratic in the translational field strength (torsion) and the Lorentz field strength (curvature).

Instead of boring you with all the details of this development to quadratic Lagrangians in a RC-spacetime and who did what and when and why, I shock you again with a messy formula. This is the most general quadratic Lagrangian including parity violating pieces (see [6]):

$$\begin{aligned}
V = & \frac{1}{2\kappa} [ (a_0 R + b_0 X - 2\Lambda) \eta \\
& + \frac{1}{3} a_2 \mathcal{V} \wedge \star \mathcal{V} - \frac{1}{3} a_3 \mathcal{A} \wedge \star \mathcal{A} - \frac{2}{3} \sigma_2 \mathcal{V} \wedge \star \mathcal{A} + a_1^{(1)} T^\alpha \wedge \star^{(1)} T_\alpha ] \\
& - \frac{1}{2\varrho} [ (\frac{1}{12} w_6 R^2 - \frac{1}{12} w_3 X^2 + \frac{1}{12} \mu_3 R X) \eta + w_4^{(4)} R^{\alpha\beta} \wedge \star^{(4)} R_{\alpha\beta} \\
& + {}^{(2)} R^{\alpha\beta} \wedge (w_2 \star^{(2)} R_{\alpha\beta} + \mu_2 {}^{(4)} R_{\alpha\beta}) + {}^{(5)} R^{\alpha\beta} \wedge (w_5 \star^{(5)} R_{\alpha\beta} + \mu_4 {}^{(5)} R_{\alpha\beta}) ] .
\end{aligned} \tag{37}$$

The first two lines represent *weak* gravity, with the conventional gravitational con-

stant  $\kappa$ , the last two lines speculative *strong* gravity with the dimensionless strong gravity constant  $\varrho$ . The unknown constants  $(a_0; a_1, a_2, a_3; b_0, \sigma_2)$ , weight the different terms of weak gravity, the unknown constants  $(w_2, w_3, w_4, w_5, w_6; \mu_2, \mu_3, \mu_4)$  those of strong gravity. What a mess!

But let us discuss the formula line by line: In the *first line*  $R$  is the EC-term,  $X := \frac{1}{4}\eta_{\alpha\beta\gamma\delta}R^{[\alpha\beta\gamma\delta]}$  is the (parity violating) curvature pseudoscalar, which vanishes in Riemannian space, but is nonvanishing in RC-space. This term is presently very popular in the quantum gravity scene,  $\Lambda$  is the cosmological constant, and  $\eta$  the ‘volume element’.

The *second line* houses all torsion-square pieces. We have a tensor torsion  ${}^{(1)}T^\alpha$ , a vector torsion  $\mathcal{V}$  and an axial vector torsion  $\mathcal{A}$ . They can enter in the combinations shown. The remarkable fact is that for dimensional reasons the first line and the second line give rise to similar effects. Instead of the EC-theory with  $R$ , you can select a suitable linear combination of torsion-square pieces acting in a RC-space with vanishing curvature (Weitzenböck space), see, for example, Itin [44] or [8] and the historical article of Sauer [45]. On the first two lines there are literally hundreds of published papers studying different properties. Numerous pages of printed pages could be saved, if our colleagues would start with the first two lines right away and just motivate their choice of the unknown constants.

Now we turn to the remaining more speculative pieces, which are, however, fairly plausible due to their Yang–Mills type structure. After all, C. N. Yang himself proposed such a theory [46]. We are not in bad company! In the *third line* we turn our attention immediately to the first three pieces: They are just squares built from the curvature scalar and/or the curvature pseudoscalar. The curvature in a RC-space decomposes into 6 irreducible pieces: they are numbered from 1 to 6. The pseudoscalar  $X$  is number 3, the scalar  $R$  number 6. The last term in the third line is then a square piece of number 4. In the *fourth line* we have the remaining curvature square pieces. The term with number 1 drops out due to certain identities.

This is only algebra. Where is the physics? you may ask. Well, we have to find out. It will be a task of the future to single out of this set of quadratic Lagrangians (37) the physically acceptable one. How such possible developments may look like, I will illustrate with one example. Shie–Nester–Yo [47] developed a fairly realistic cosmological model of Friedmann type with *propagating connection* by



picking the Lagrangian

$$V_{\text{SNY}} = \underbrace{\frac{1}{2\kappa} (a_0 R \eta + a_1 {}^{(1)}T^\alpha \wedge {}^{*(1)}T_\alpha)}_{\text{weak Newton-Einstein gravity}} - \underbrace{\frac{w_6}{24\varrho} R^2 \eta}_{\text{strong YM-type gravity}}. \quad (38)$$

They found two conventional graviton helicities, as in GR, and this, for  $a_0 \neq a_1$ , combined with a torsion mode of mass of  $\mu := a_1 - a_0$  and spin  $0^+$ , which has many attractive features. Of course, equation (38) is a subcase of equation (37). In the meantime this paper has been generalized by including parity violating pieces, inter alia, and it has been numerically evaluated. This paper has about 20 follow-up papers. In this way one collects more and more insight into the possible physics behind the most general quadratic PG-Lagrangian.

## 12 Outlook

What is the benefit of all of that for the theory of spacetime? Well, it is a small but decisive step beyond the established Riemannian spacetime structure of GR. Cartan's torsion has been incorporated into the body of knowledge of classical spacetime geometry. At the same time it has been demonstrated that the Poincaré group  $P(1, 3) = T(4) \rtimes SO(1, 3)$ , acting in the Minkowski space, and the behavior of the Dirac field in non-inertial frames leads, via the gauge principle, to the Riemann–Cartan geometry of spacetime. That is, the  $P(1, 3)$  symmetry induced the Riemann–Cartan geometry.

The generalization of this procedure seems to be straightforward. If we add the group of dilations to  $P(1, 3)$ , assuming scale covariance in addition to the  $P(1, 3)$  covariance, we arrive at the 11 parametric Weyl group. Gauging it, requires one more potential, namely the Weyl covector  $Q$ , defined in terms of the nonmetricity according to  $Q := \frac{1}{4}Q_\gamma{}^\gamma = \frac{1}{4}g^{\alpha\beta}Q_{\alpha\beta}$ , see equation (12). Associated with it comes a conservation law and the Noether current  $\Delta = \delta L/\delta Q$ , the dilation or scale current, which Weyl had mistaken for the electric current. If we turn the crank, a Weyl–Cartan spacetime emerges together with a gauge field equation that has the dilation current as source. This is standard Weyl lore from a contemporary point of view, see [8], Chapter 8.

I hope it doesn't take you by surprise that I cannot see much common ground with the theory of E. Scholz presented during this workshop [32]. In his approach, spacetime is governed by a Weyl geometry with vanishing torsion, but the dilation current is not an inhabitant of the Weyl space of Scholz—or, at least, this

current has not been identified as such and lives anonymously and drifts around uncontrolled by any field equation.

Instead, one can add to the  $P(1, 3)$  simple supersymmetry (symmetry between fermions and bosons) by extending the Poincaré algebra with anticommuting fermionic generators thus being led to a Poincaré superalgebra. The corresponding gauge procedure creates a so-called superspace(time) geometry. The field equations of simple supergravity can be immediately written down by using the EC-field equations (34) and (35); as sources one takes the energy-momentum and the spin currents of the massless Rarita–Schwinger field, which carries spin  $\frac{3}{2}$ . The Rarita–Schwinger field conspires with the effective spin 2 of the EC-field to build up a super multiplett  $(2, \frac{3}{2})$ , compare [8], Chapter 12.

In this way we see that also in supersymmetry the gauge concept of Weyl and Yang–Mills–Utiyama is successful. And the geometry of spacetime turned out to have a potential “super” structure beyond Riemann–Cartan geometry.

## Acknowledgments

I would like to thank Dennis Lehmkuhl, Erhard Scholz, and Gregor Schiemann most sincerely for the invitation to this workshop, for the hospitality, and for the lively discussions. I am also grateful to Peter Baekler (Düsseldorf), Milutin Blagojević (Belgrade), Yakov Itin (Jerusalem), Bahram Mashhoon (Columbia, Missouri), Eckehard Mielke (Mexico City), James Nester (Chung-li), Yuri Obukhov (Cologne/Moscow), and Dirk Pützfeld for many oral or email discussion on the subject of gravity and gauging. This work was supported by the German-Israeli Foundation for Scientific Research and Development (GIF), Research Grant No. 1078-107.14/2009.

## References

- [1] J. Ehlers and C. Lämmerzahl (eds.), *Special Relativity. Will it survive the next 100 years?* (Springer, Berlin, 2006). 3
- [2] B. Mashhoon, *Necessity of acceleration-induced nonlocality*, *Annalen der Physik* (Berlin) **523**, 226–234 (2011) [[arXiv:1006.4150](https://arxiv.org/abs/1006.4150)]. 3
- [3] B. Famaey et al. (organizers), *Modified Gravity Approaches to the Dark Sector*, Conference at the Observatoire Astronomique de Strasbourg, 28 June to 1 July 2010; for the talks see 3  
[<http://astro.u-strasbg.fr/MGAtotheDARK/Home.html>].
- [4] C. Kiefer, *Quantum Gravity*, 2nd ed. (Oxford University Press, Oxford, UK, 2007). 4

- [5] D. Diakonov, A. G. Tumanov and A. A. Vladimirov, *Low-energy General Relativity with torsion: A Systematic derivative expansion*, Phys. Rev. D **84**, 124042 (2011) [16 pages] [[arXiv:1104.2432](#)]. 4
- [6] P. Baekler and F. W. Hehl, *Beyond Einstein–Cartan gravity: Quadratic torsion and curvature invariants with even and odd parity including all boundary terms*, Class. Quant. Grav. 28, 215017 (2011) [11 pages] [[arXiv:1105.3504](#)]. 4, 23
- [7] Y. N. Obukhov and F. W. Hehl, *Extended Einstein–Cartan theory à la Diakonov: the field equations* [13 pages] [[arXiv:1202.6045](#)]. 4
- [8] M. Blagojević and F. W. Hehl (eds.), *Gauge Theories of Gravitation*, a reader with commentaries (Imperial College Press, London, in press, 2012). 4, 7, 13, 15, 21, 22, 24, 25, 26, 27, 30
- [9] R. Utiyama, *Invariant theoretical interpretation of interactions*, Phys. Rev. **101**, 1597–1607 (1956). 5
- [10] T. W. B. Kibble, *Lorentz invariance and the gravitational field*, J. Math. Phys. **2**, 212–221 (1961). 6, 10, 21
- [11] T. W. B. Kibble, Excerpt from the Foreword in [8]. 6
- [12] A. Einstein: *The Meaning of Relativity*, 5th ed. (Princeton University Press, Princeton, NJ, 1955) [first published in 1922]. 7
- [13] A. Einstein, Reprint 1.1 in [8]. 9
- [14] R. Adler, M. Bazin, and M. Schiffer, *Introduction to General Relativity*, 2nd ed. (McGraw-Hill, New York, 1975). 9
- [15] H. Weyl, *Space-Time-Matter*, translated from the fourth German edition of 1921 by H. Brose (Dover, New York, 1952). 9
- [16] R. Colella, A.W. Overhauser, and S.A. Werner: *Observation of gravitationally induced quantum interference*, Phys. Rev. Lett. **34**, 1472–1474 (1975). 10
- [17] H. Rauch and S.A. Werner, *Neutron Interferometry*, Lessons in Experimental Quantum Mechanics (Clarendon Press, Oxford, UK, 2000). 10

- [18] U. Bonse and T. Wroblewski, *Measurement of neutron quantum interference*, Phys. Rev. Lett. **51**, 1401–1404 (1983). 10
- [19] F.W. Hehl and Yu.N. Obukhov, *Foundations of Classical Electrodynamics: Charge, Flux, and Metric* (Birkhäuser, Boston, MA, 2003). 11, 14
- [20] J. A. Schouten, *Tensor Analysis for Physicists*, 2nd ed. reprinted (Dover, Mineola, NY, 1989). 12, 15
- [21] A. Einstein, Excerpt from the Preface (dated 04 April 1955) in: *Cinquant’anni di Relatività 1905–1955*. M. Pantaleo (ed.) (Edizioni Giuntine and Sansoni Editore, Florence, 1955) (translation from the German original by F. Gronwald, D. Hartley, and F. W. Hehl). 12
- [22] H. Weyl, *Elektron and Gravitation. I*, Zeitschrift für Physik **56**, 330–352 (1929), translation into English in [37], pp. 121–144. 13
- [23] H. Weyl, *A remark on the coupling of gravitation and electron*, Phys. Rev. **77**, 699–701 (1950). 13
- [24] A. Einstein, *Einheitliche Feldtheorie von Gravitation und Elektrizität*, Sitzungsber. Preuss. Akad. Wiss. Berlin, Phys.-math. Klasse (1925) pp. 414–419. 13
- [25] E. Schrödinger, *Space-Time Structure*, reprinted with corrections (Cambridge University Press, Cambridge, UK, 1960). 14
- [26] F. W. Hehl and Y. N. Obukhov, *Spacetime metric from local and linear electrodynamics: A new axiomatic scheme*, Lect. Notes Phys. (Springer) **702**, 163–187 (2006) [arXiv:gr-qc/0508024]. 14
- [27] F. Gronwald and F. W. Hehl, *On the gauge aspects of gravity*, in: *Proc. Int. School of Cosm. & Gravit. 14<sup>th</sup> Course: Quantum Gravity*. Held in Erice, Italy. Proceedings, P.G. Bergmann et al. (eds.) (World Scientific, Singapore, 1996) p. 148 [arXiv:gr-qc/9602013]. 15
- [28] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Ne’eman, *Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance*, Phys. Rept. **258**, 1–171 (1995). 15, 19, 20

- [29] D. Hartley: *Normal frames for non-Riemannian connections*. Class. Quantum Grav. **12** (1995) L103–L105 [[arXiv:gr-qc/9510013](#)]. 16
- [30] J. M. Nester, *Normal frames for general connections*, Annalen der Physik (Berlin) **19**, 45–52 (2010). 16
- [31] J. F. T. Giglio and W. A. Rodrigues, Jr, *Locally inertial reference frames in Lorentzian and Riemann–Cartan spacetimes* [20 pages] [[arXiv:1111.2206](#)]. 17
- [32] E. Scholz, *Paving the way for transitions between theories—a case for Weyl geometry*, this workshop, manuscript [32 pages] (March 2012). 17, 25
- [33] P. von der Heyde, *The equivalence principle in the  $U_4$  theory of gravitation*, Nuovo Cim. Lett. **14**, 250–252 (1975). 17
- [34] F.W. Hehl and W.-T. Ni, *Inertial effects of a Dirac particle*, Phys. Rev. D **42**, 2045–2048 (1990). 17
- [35] E. M. Corson, *Introduction to Tensors, Spinors, and Relativistic Wave-Equations* (Blackie, London, 1953). 19
- [36] D. W. Sciama, *On the analogy between charge and spin in general relativity*, in: *Recent Developments in General Relativity*, Festschrift for Infeld (Pergamon Press, Oxford, UK; PWN, Warsaw, 1962), pp. 415–439. 21
- [37] L. O’Raifeartaigh, *The Dawning of Gauge Theory* (Princeton University Press, Princeton, NJ, 1997). 21, 28
- [38] M. Blagojević, *Gravitation and Gauge Symmetries* (IoP Publishing, Bristol, UK, 2002). 21
- [39] L. Ryder, *Introduction to General Relativity* (Cambridge University Press, Cambridge, UK, 2009). 21
- [40] F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. M. Nester, *General relativity with spin and torsion: Foundations and prospects*, Rev. Mod. Phys. **48**, 393–416 (1976). 22
- [41] A. Trautman, *The Einstein–Cartan theory*, in: *Encyclopedia of Mathematical Physics*, vol. 2, J.-P. Francoise et al. (eds.) (Elsevier, Oxford, UK, 2006) pp. 189–195. 22

- [42] Y. Mao, M. Tegmark, A. Guth and S. Cabi, *Constraining torsion with Gravity Probe B*, Phys. Rev. **D76**, 104029 (2007) [26 pages] [[arXiv:gr-qc/0608121](#)]. 23
- [43] D. Puetzfeld and Y. N. Obukhov, *Probing non-Riemannian spacetime geometry*, Phys. Lett. A **372**, 6711–6716 (2008) [[arXiv:0708.1926](#)]. 23
- [44] Y. Itin, *Energy-momentum current for coframe gravity*, Class. Quantum Grav. **19**, 173–189 (2002) [[arXiv:gr-qc/0111036](#)]. 24
- [45] T. Sauer, *Field equations in teleparallel spacetime: Einstein’s Fernparallelismus approach towards unified field theory*, Historia Math. **33**, 399–439 (2006) [[arXiv:physics/0405142](#)]. 24
- [46] C. N. Yang, *Integral formalism for gauge fields*, Phys. Rev. Lett. **33**, 445–447 (1974). Reprint 19.1 in [8]. 24
- [47] K. F. Shie, J. M. Nester, and H. J. Yo, *Torsion cosmology and the accelerating universe*, Phys. Rev. D **78**, 023522 (2008) [16 pages] [[arXiv:0805.3834](#)]. 24